COMPLEXITY, SCALING, AND A PHASE TRANSITION

Jiayue Yang, and Andrew R. Frey (JHEP09(2023)029)

Contact j43yang@uwaterloo.ca (from Robert Mann and Niayesh Afshordi's group)

I: Background Information

- Complexity measures the difficulty to do a task.
- Complexity is defined as the minimal number of quantum gates in states transformations.

$$\psi = U\psi_0 = g_n g_{n-1} \dots g_1 \psi_0 \tag{1}$$

- Complexity is an important concept to understand quantum information in gravity.
  - In AdS/CFT, holographic complexity is the gravity dual of the complexity of a boundary theory.
  - This boundary complexity is a measure of the distance in Hilbert space between states.

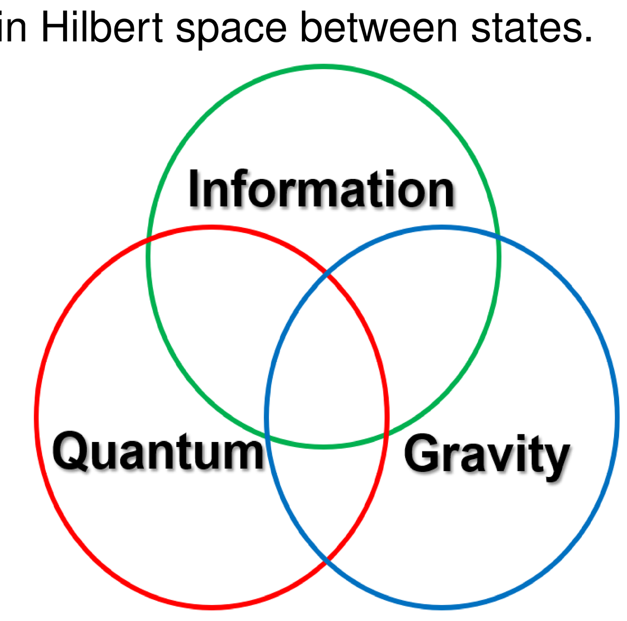

Fig1: Quantum, information, and gravity.

II: Holographic Complexity

Motivated by the black hole physics in AdS/CFT, many holographic proposals have been put forward.

1. Complexity=Volume 1: the complexity of the boundary CFT state is proportional to the volume of the maximal volume slice **(Susskind and Stanford 2014)**.
2. Complexity=Volume 2: the complexity of the boundary CFT state is proportional to the spacetime volume of the Wheeler-DeWitt patch **(Couch, Fischler, and Nguyen 2016)**.
3. Complexity=Action: the complexity of the boundary CFT state is proportional to the action of the Wheeler-DeWitt (WDW) patch **(Brown, Roberts, Susskind, Swingle, and Zhao 2015)**.

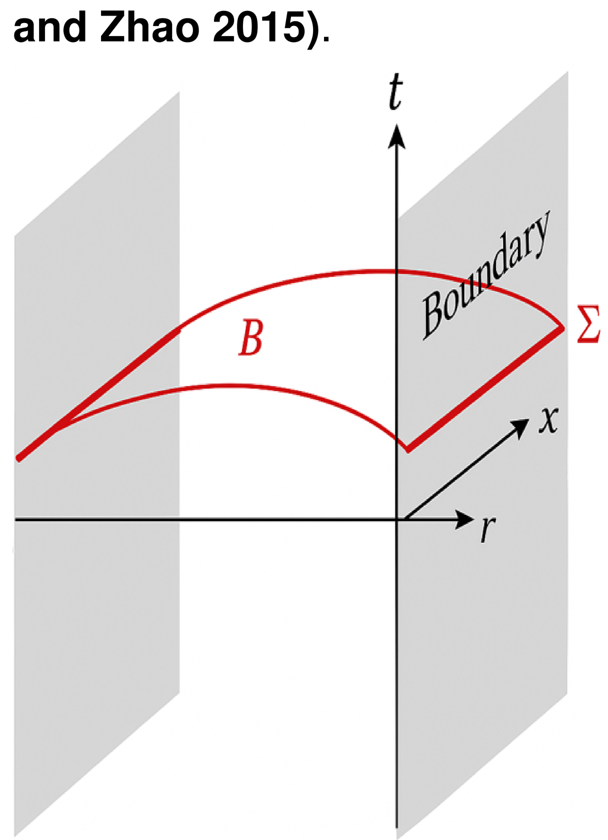

Fig2: Sketch of Complexity=Volume proposal.

The WDW patch is the spacetime region bounded by future- and past-directed lightsheets emitted from the boundary time slice where the complexity is measured.

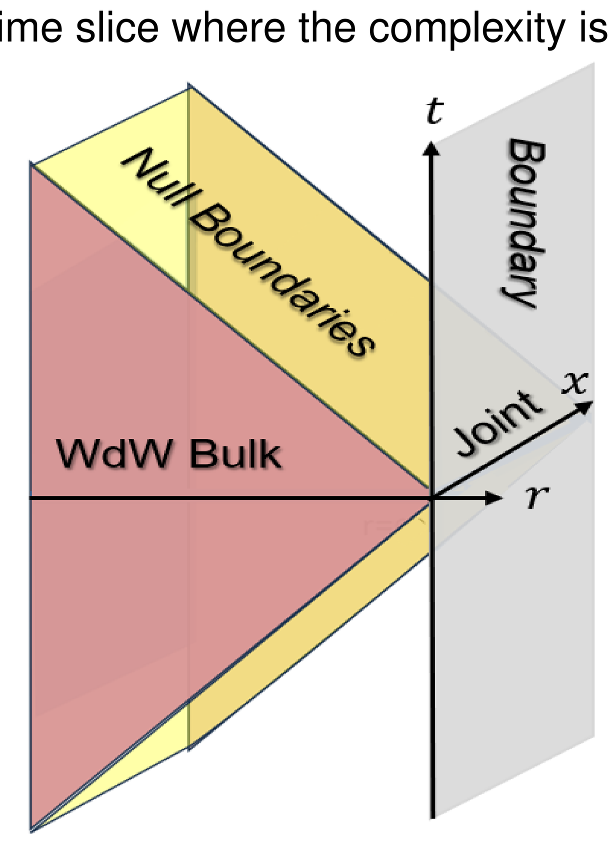

Fig3: Sketch of Complexity=Action proposal.

Complexity of formation is defined as complexity of the magnetized AdS solitons subtract the corresponding complexity of periodic AdS.

III: Magnetized AdS Solitons

Suppose that the gauge theory we consider is on a flat spacetime with one spatial coordinate ϕ periodically identified with period $\Delta\phi$. In this case, the gauge theory can have a $U(1)$ Wilson line around the ϕ direction, and there are three regular solutions. The first solution is AdS with a periodic direction and a Wilson line,

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + d\phi^2\right) + \frac{l^2}{r^2}dr^2, \quad A = -\frac{\Phi}{\Delta\phi}d\phi \tag{2}$$

for holonomy Φ of the boundary gauge field. The other two solutions are generalizations of the AdS soliton **(Witten, Horowitz and Myers 1998)**; both have metric (for $d = 3, 4$)

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \tag{3}$$

$$f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}} \tag{4}$$

and gauge field

$$A = \sqrt{7-d}\,Q\left(\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}}\right)d\phi, Q \equiv \frac{1}{\sqrt{7-d}}\frac{r_0^{d-2}\Phi}{\Delta\phi}. \tag{5}$$

The periodic direction of the magnetized solitons shrinks smoothly at a finite AdS radius r_0, no horizon!

IV: Complexity=Volume 1

The initial proposal for complexity, known as CV or "complexity=volume", for an asymptotically AdS spacetime is $C_V = (d-1)V/2\pi^2 Gl$, where V is the volume of a maximal volume slice anchored at a fixed time on the boundary. For both the magnetized solitons and periodic AdS, this volume is

$$V = V_{\vec{x}}\Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-2} = \frac{V_{\vec{x}}\Delta\phi}{d-1}\frac{r_m^{d-1} - r_0^{d-1}}{l^{d-2}}, \tag{6}$$

with $r_0 \to 0$ for periodic AdS, where $V_{\vec{x}}$ is the volume along the $\vec{x}$ directions.

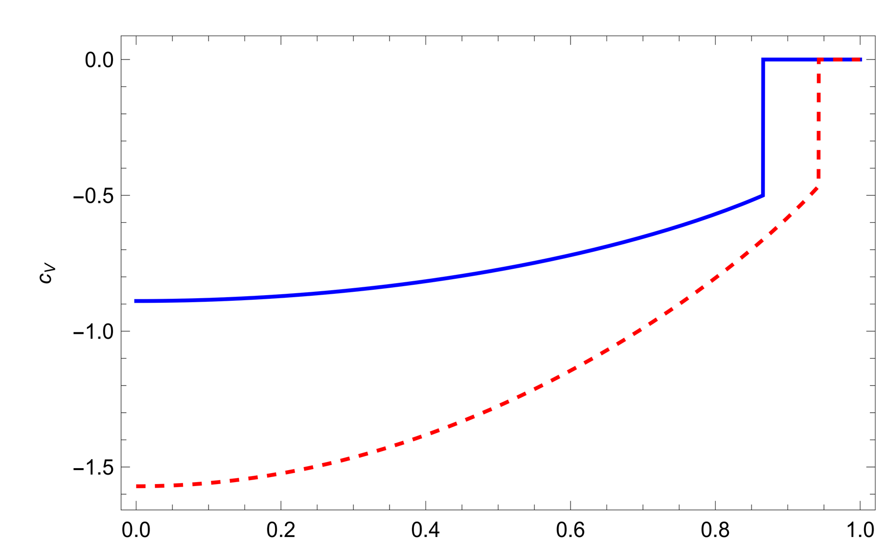

Fig4: CV complexity of formation per unit volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_V \equiv G\mathcal{C}_V(\Delta\phi/l)^{d-1}$.

V: Complexity=Volume 2

Another proposal for holographic complexity, known as CV2.0, is that $C_2 \equiv V_{WDW}/Gl^2$, where V_{WDW} is the spacetime volume of the Wheeler–DeWitt (WDW) patch. The WDW patch volume is

$$V_{WDW} = \int d^{d+1}x\sqrt{-g} = 2V_{\vec{x}}\Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-1} t_F(r) \tag{7}$$

where

$$t_F(r) = \frac{l^2}{r_0}\int_{\tilde{r}}^{r_m/r_0} \frac{d\tilde{r}'}{\tilde{r}'^2\sqrt{f(\tilde{r}')}} \equiv \frac{l}{r_0}\tilde{r}_F(\tilde{r}). \tag{8}$$

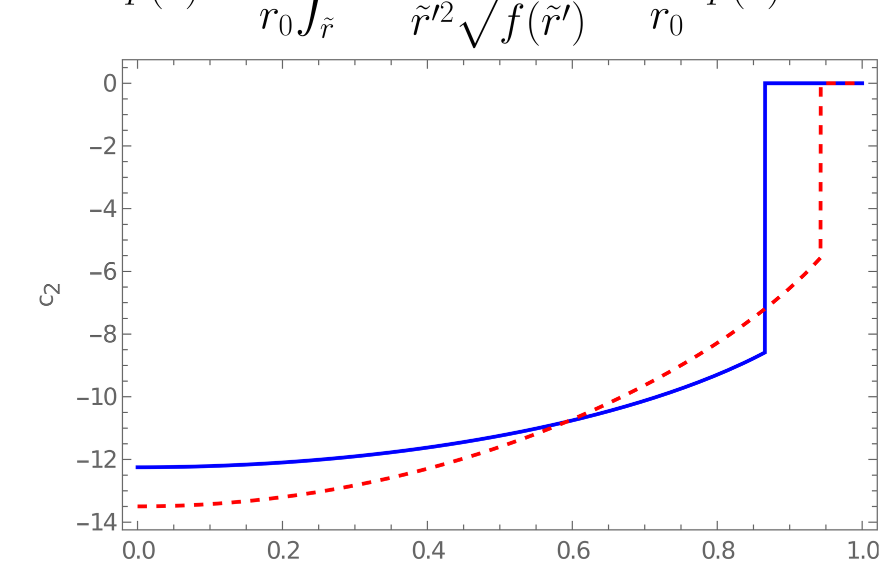

Fig5: CV2.0 complexity of formation per boundary volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_2 \equiv G\mathcal{C}_2(\Delta\phi/l)^{d-1}$.

VI: Complexity=Action

The action complexity ("complexity=action" or CA) is given by $C_A = S_{WDW}/\pi$, where S_{WDW} is the action evaluated on the WDW patch, including appropriate terms on the boundary of the patch. Altogether, this action is

$$S_{WDW} = S_{bulk} + S_{bdy} + S_{joint}. \tag{9}$$

And we can include an additional Maxwell boundary term $(2\nu - 1)/4 \times F_{\mu\nu}F^{\mu\nu}$ in the bulk action .

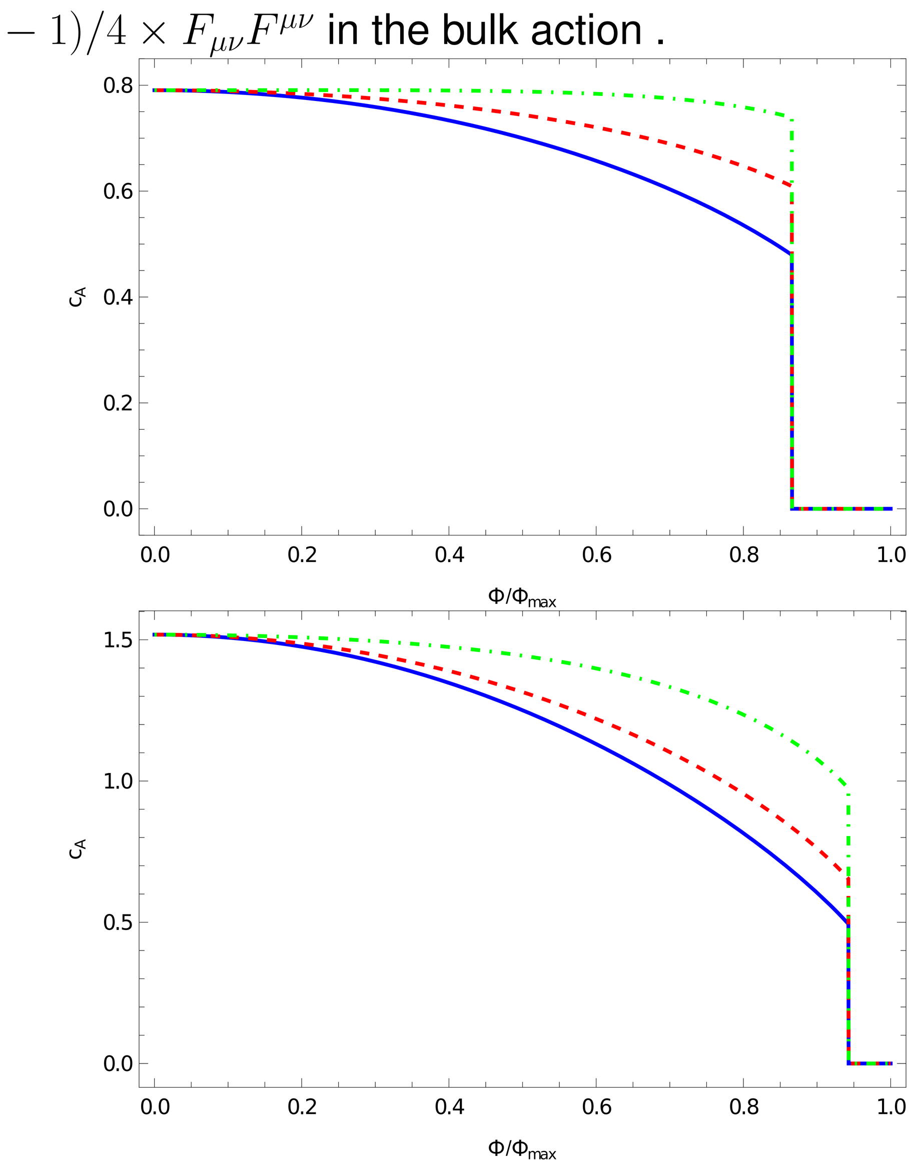

Fig6: CA complexity of formation per unit boundary volume for $d = 3, 4$ as labeled with $\nu = 0$ (solid blue), $\nu = 1/(d-1)$ (dashed red), and $\nu = 1$ (dot-dashed green). Plotted curves are $c_A = G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of the Wilson line Φ.

VII: Phase Transition

- There is a transition from the magnetized soliton (confining, no horizon) phase to the periodic AdS (deconfined, with horizon) phase.
- The complexity of formation acts like an order parameter, vanishing in the deconfined phase and changing discontinuously at the transition.

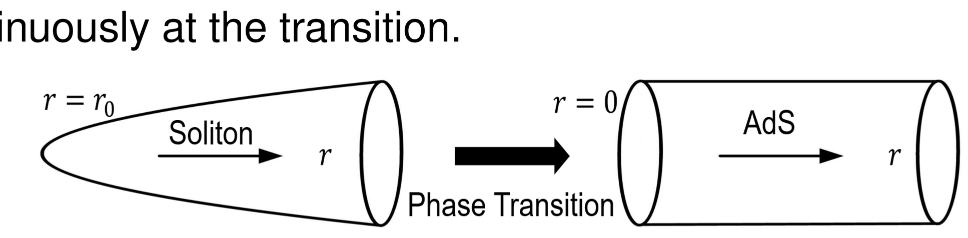

Fig7: Magnetized Soliton- AdS Phase transition

VIII: Scaling Properties

- Striking result: the complexity of formation density for magnetized AdS solitons scales as the inverse $(d-1)$st power of the circumference of the boundary circle, i.e.,

$$c \propto 1/\Delta\phi^{d-1} \tag{10}$$

- General holographic complexity functional advocated by the "complexity=anything" program **(Belin, Myers, Ruan, Sárosi, and Speranza 2022)** also exhibits the same scaling law.

IX: Outlook

- Future directions
  - **Relativistic Quantum Information**
  - **Holographic Black Hole Thermodynamics**
- Acknowledgments: JY would like to thank Robert Mann, Niayesh Afshordi, Haijun Wang for encouragement, thank Chuanxin Cui, Wencong Gan, Ming Zhang, Yusen An, and Shanming Ruan for helpful disccussions.

Feel free to scan the QR code and visit my website https://link3.cc/jiayue! I'm open to collaborations!